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Tensor Products in Generalized Measure Theory

Alexander Wilce[1]

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Kläy, Randall, and Foulis established that the signed weight space of the tensor product of two quasimanuals each having a positive, finite-dimensional state space is isomorphic to the algebraic tensor product of the signed-weight spaces of the factors. We obtain a generalization of this result for arbitrary quasimanuals. A compactness condition due to Cook—here called *discreteness*—is discussed and shown to be preserved under the formation of tensor products. It is shown that the predual of the signed weight space of a tensor product of discrete manuals is the projective (ordered) tensor product of the preduals of the signed weight spaces of the factors.

1. INTRODUCTION

Over the last several decades, and especially since the work of Mackey (1965), many attempts have been made to frame a generalized measure or probability theory broad enough to accommodate quantum mechanics as a special case, while avoiding any ad hoc imposition of linear or *-algebraic structure on the space of observables. These include the theory of measures on orthomodular lattices and posets (Gudder, 1988), various theories considering convex sets as abstract state spaces (e.g., Mielnik, 1968, Davies, 1976), and the theory of states or stochastic functions on hypergraphs or manuals (Foulis and Randall, 1981).

A serviceable generalized measure theory ought to provide a device—let us speak of a "tensor product"—whereby a model of a complex system comprised of noninteracting components can be constructed from the models of the components. In classical probability theory, one forms products of σ-fields; in orthodox quantum mechanics, one forms tensor products of Hilbert spaces. The matter is problematic in more general theories: Various accounts suffer from either the absence of, or the proliferation of candidates

[1]Department of Mathematics, University of Pittsburgh/Johnstown, Johnstown, Pennsylvania 15904.

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for, such a product. For instance, in Foulis and Randall (1981) it is shown that the category of orthomodular posets admits no reasonable tensor product; on the other hand, there are at least two natural tensor products for convex sets and ordered Banach spaces (Namioka and Phelps, 1969; Wittstock, 1974)—the *injective* and the *projective* tensor product. These coincide in the case of classical measure theory, but are in general quite different. Neither corresponds to the tensor product in quantum mechanics.

In contrast, there does exist a heuristically and technically natural tensor product for quasimanuals (Foulis and Randall, 1981; Lock, 1981) (for the definition, see Section 3). This tensor product is well-behaved for quasimanuals with finite-dimensional state spaces in the following sense. If $\mathscr{A}$ denotes a quasimanual, denote by $V(\mathscr{A})$ the *signed weight space* of $\mathscr{A}$, i.e., the linear hull of $\mathscr{A}$'s state space. $\Omega(\mathscr{A})$ is said to be *positive* iff for all $x \in X$, there exists a state $\omega \in \Omega$ such that $\omega(x) > 0$.

Theorem 1.1 (Kläy, Randall, and Foulis, 1987). Let $\mathscr{A}$ and $\mathscr{B}$ be quasimanuals with positive, finite-dimensional state spaces, and let $\mathscr{A}\mathscr{B}$ be their (pre-) tensor products. Then $V(\mathscr{A}\mathscr{B}) \simeq V(\mathscr{A}) \otimes V(\mathscr{B})$.

In what follows we extend this result in two ways. First, we prove a straightforward lemma characterizing $V(\mathscr{A}\mathscr{B})$, for *arbitrary* quasimanuals $\mathscr{A}$ and $\mathscr{B}$, as the space of regular weak*-to-weak continuous operators from $V^*(\mathscr{A})$ to $V(\mathscr{B})$. As an immediate corollary, we recover Theorem 1.1, less the hypothesis of positivity. Second, we show that if $V(\mathscr{A})$ and $V(\mathscr{B})$ are the duals, respectively, of the linear spans $L_o(\mathscr{A})$ and $L_o(\mathscr{B})$ of the evaluation functionals associated with outcomes of $\mathscr{A}$ and $\mathscr{B}$, then $\mathscr{A}\mathscr{B}$ again has this property, and

$$L_o(\mathscr{A}\mathscr{B}) = L_o(\mathscr{A}) \otimes_p L_o(\mathscr{B})$$

where the tensor product on the right is the projective tensor product of ordered normed spaces.

We begin with some background material on signed weight spaces. The exposition derives from the paper of Cook (1985); certain results of that paper are extended. We suppose the reader to be familiar with the rudiments of the theory of ordered linear spaces, as outlined, e.g., in Alfsen (1971), and also with basic manual-theoretic notions, as outlined, for instance, in Gudder (1988). Throughout this paper, we denote the positive cone of an ordered vector space V by V_+, and the positively-generated part of V, i.e., the space $V_+ - V_+$ of regular elements of V, by V^+. The extreme boundary of a convex set Δ is denoted by $\partial\Delta$.

2. SPACES OF SIGNED WEIGHTS

Let $\mathscr{A}$ be a quasimanual with outcome set $X(\mathscr{A})$ and state space $\Omega(\mathscr{A})$. A *signed weight* on $\mathscr{A}$ is a linear combination (in $\mathbf{R}^X$) of states; the space of all such—the *signed weight space* of $\mathscr{A}$—is denoted by $V(\mathscr{A})$. Recall that if $\mathscr{A}$ is a premanual, then every state on $\mathscr{A}$ extends uniquely to the manual closure $\langle\mathscr{A}\rangle$; hence $V(\mathscr{A}) = V(\langle\mathscr{A}\rangle)$. The state space of $\mathscr{A}$ is a base for the positive cone of $V(\mathscr{A})$ [ordered pointwise on $X(\mathscr{A})$]. It can be shown (Cook, 1985) that the base-normed space $(V(\mathscr{A}), \Omega(\mathscr{A}))$ is always complete. It follows that the dual $V^*(\mathscr{A})$ of a signed weight space is a complete order-unit normed space.

To every event A of $\mathscr{A}$ there is associated an element f_A of the order interval $[0, e]$ of V^* given by $f_A(\omega) = \sum_{x\in A} \omega(x)$. We write f_X for $f_{\{x\}}$. If $E \in \mathscr{A}$, then $f_E = e$. We will denote the span in $V^*(\mathscr{A})$ of the functionals f_A by $L(\mathscr{A})$, and the span of those functionals f_A arising from *finite* events by $L_o(\mathscr{A})$. Note that $L(\mathscr{A})$ is an order-unit normed space, while $L_o(\mathscr{A})$ is generally not.

Example 2.1. (1) If $\mathscr{A}$ consists of a single set E, then $V(\mathscr{A}) \simeq l_1(E)$. Now, $L(\mathscr{A})$ consists of the functions on E having finite range; hence, $L(\mathscr{A})$ is dense in $V^* = l^\infty(E)$. $L_o(\mathscr{A})$ consists of the finitely nonzero functions on E, hence $\bar{L}_o(\mathscr{A}) \simeq c_o(E)$. (2) Let (S, Σ) be a measurable space. The associated *partition manual* $\mathscr{A}(S, \Sigma)$ consists of all countable partitions of S by nonempty Σ-measurable sets. $\Omega(\mathscr{A})$ consists of the σ-additive probability measures on (S, Σ) and $V(\mathscr{A})$ is the space of all σ-additive measures on (S, Σ). $L_o(\mathscr{A})$ and $L(\mathscr{A})$ coincide and may be identified with the space of simple Σ-measurable functions. (3) Let μ be a σ-additive probability measure on (S, Σ). Let $\mathscr{A}_\mu$ be the collection of countable partitions of the identity in the measure algebra Σ/μ. We can identify $\Omega(\mathscr{A}_\mu)$ with the convex set of μ-absolutely continuous probability measures on (S, Σ); hence $V(\mathscr{A}) \simeq L^1(\mu)$. Now, $L_o(\mathscr{A}) = L(\mathscr{A})$ corresponds to the space of simple functions in $L^\infty(\mu)$. (4) Let $\mathbf{H}$ be a Hilbert space (real or complex). The associated *frame manual* $\mathscr{F}(\mathbf{H})$ consists of the maximal orthonormal subsets of $\mathbf{H}$. If $\dim(\mathbf{H}) > 2$, Gleason's theorem provides an isomorphism between $V(\mathscr{F})$ and the self-adjoint trace-class of $\mathbf{H}$. We can identify $L_o(\mathscr{F})$ with the space of finite-rank self-adjoint operators on $\mathbf{H}$ and its closure [in $V^*(\mathscr{F})$] with the space of self-adjoint compact operators on $\mathbf{H}$.

In certain cases, it is reasonable to assume that the set of states of a physical system is compact in some locally convex topology. In the classical setting, where the state space is the convex set of regular Borel probability measures on a locally compact Hausdorff space S, the topology one has in mind is the relative weak-* topology inherited from $\mathscr{C}(S)^*$ by the usual identification of such measures with Radon measures. In contrast, the weakest topology naturally supplied by the construction of an arbitrary signed-weight space $V(\mathscr{A})$ is the relative product topology inherited from $\mathbf{R}^{X(\mathscr{A})}$; a slightly stronger topology is that of eventwise convergence. In the setting of

Example 2.1(2) above, these are the same, and correspond to the weakest topology making simple measurable functions continuous (as functionals); this coincides with the topology induced by the continuous functions iff the Hausdorff space in question is a finite set with its discrete topology. Indeed, the following has been observed by Cook:

Lemma 2.2. The following are equivalent: (1) Ω is eventwise compact; (2) Ω is pointwise compact; (3) $\mathscr{A}$ is locally finite.

If Ω is compact in a given locally convex topology, so is the unit ball of V. Hence if $\mathscr{A}$ is locally finite, then $V(\mathscr{A})$ is the dual of $L_o(\mathscr{A})$. The converse is false [consider Example 2.1(1)]. The condition that $V(\mathscr{A})$ be the dual of $L_o(\mathscr{A})$ was studied by Cook (1985). We propose to call a quasi-manual $\mathscr{A}$ satisfying this condition *discrete*. Thus, any locally finite quasi-manual is discrete, as is the classical manual $\mathscr{A}=\{E\}$ of Example 2.1(1). Also, the frame manual of a separable Hilbert space **H** [Example 2.1(4)] is discrete, since **H**'s self-adjoint trace class is the dual of the space of finite-rank self-adjoint operators on **H**.

Lemma 2.3. Let V be a base-normed space with positive cone C and unit ball U. If τ is any linear topology on V in which C is closed, then $C \cap U$ is τ-compact iff U is τ-compact.

Proof. Trivially, (1) implies (2). Conversely, suppose the positive part $C \cap U$ of V's unit ball is τ-compact. Let x_λ be a net in the unit ball. Then (Alfsen, 1971, Proposition II.3.1) there can be found for each λ positive, elements y_λ and z_λ of V with $x_\lambda = y_\lambda - z_\lambda$ and $\|x_\lambda\| = \|y_\lambda\| + \|z_\lambda\|$—in particular, y_λ and z_λ belong to $C \cap U$. Choose τ-convergent subnets $y_{\lambda'}$ and $z_{\lambda'}$ (identically indexed), and observe that the subnet $x_{\lambda'}$ is also τ-convergent to an element of U. ■

Discreteness guarantees that Ω has many extreme points (i.e., pure states). The following simple application of Choquet theory allows one to recover the remaining states as "mixtures" of the pure states.

Proposition 2.4. Let $\mathscr{A}$ be discrete and contain at least one countable operation. Then $\partial\Omega$ can be made into a measurable space in such a way that for every $\omega \in \Omega$ there exists a probability measure m on $\partial\Omega$ such that for all $x \in X(\mathscr{A})$,

$$\omega(x) = \int_{\partial\Omega} v(x)\, dm(v)$$

Proof. Let Δ denote the positive part of the closed unit ball of $V(\mathscr{A})$. Let $\mathscr{B}_o$ denote the trace on $\partial\Delta$ of the field of Baire sets, i.e., the σ-field generated by sets $f^{-1}(0) \cap \partial\Delta$ where f is a continuous real-valued functional on Δ. The Bishop–DeLeuw theorem (Alfsen, 1971, Theorem I.4.14) yields for every $\omega \in \Delta$ a probability measure m on $\mathscr{B}_o$ such that for all continuous affine functionals f on Δ, $f(\omega) = \int_{\partial\Delta} f(v)\, dm(v)$. Let $\omega \in \Omega(\mathscr{A}) \subseteq \Delta$; If $f = f_x$ for some x in X, we obtain $\omega(x) = \int_{\partial\Delta} v(x)\, dm(v)$. Now, $\partial\Omega$ is $\partial\Delta$ less the single extreme point 0. Let $E = \{x_n | n \in \mathrm{N}\}$ be a countable operation in $\mathscr{A}$, and let $f_n = f_{\{x_1,\ldots,x_n\}}$ for $n = 1, 2, \ldots$. Then f_n is continuous on Δ, and hence $\{0\} = \bigcap_n f_n^{-1}(0)$ is a Baire set. Hence, $\partial\Omega$ is an element of $\mathscr{B}_o$, whence

$$\int_{\partial\Delta} f_n(v)\, dm(v) = \int_{\partial\Omega} f_n(v)\, dm\,(v) + \int_{\{0\}} f_n(v)\, dm(v) = \int_{\partial\Omega} f_n(v)\, dm(v)$$

Finally, since $\lim_n f_n(\omega) = 1$, we have $m(\{0\}) = 0$, i.e., $m(\partial\Omega) = 1$. ■

3. THE SIGNED WEIGHT SPACE OF A TENSOR PRODUCT

The following constructions were introduced by Foulis and Randall (1981). Given a pair of quasimanuals $\mathscr{A}$ and $\mathscr{B}$, define a compound operation as follows: First, execute a given $E \in \mathscr{A}$; obtaining the outcome $x \in E$, execute a preselected operation $E_x \in \mathscr{B}$. Upon obtaining outcome $y \in F_x$, record the ordered pair (x, y) as the outcome of the compound experiment. If we adopt the notation xy for an ordered pair, writing AB for $A \times B$ and xA for $\{x\} \times A$, the sample space for such a compound experiment is $\bigcup_{x \in E} xF_x$. The collection of such experiments forms a quasimanual denoted by $\overrightarrow{\mathscr{A}\mathscr{B}}$. Notice that $\bigcup \overrightarrow{\mathscr{A}\mathscr{B}} = XY$, where X and Y are the outcome sets of $\mathscr{A}$ and $\mathscr{B}$, respectively. Call a compound experiment whose second component is independent of the outcome of the first—i.e., a set EF—*a product operation*. The collection of product operations is a subquasimanual of $\overrightarrow{\mathscr{A}\mathscr{B}}$; if the notational abuse may be forgiven, we will denote it by $\mathscr{A} \times \mathscr{B}$. Note that since $\bigcup (\mathscr{A} \times \mathscr{B}) = XY$, states on $\overrightarrow{\mathscr{A}\mathscr{B}}$ are states on $\mathscr{A} \times \mathscr{B}$. The proof of the following is straightforward:

Lemma 3.1. Let $\omega \in \Omega(\mathscr{A} \times \mathscr{B})$. Then $\omega \in \Omega(\overrightarrow{\mathscr{A}\mathscr{B}})$ if and only if $\omega(xF) := \sum_{y \in F} \omega(xy)$ is independent of the choice of $F \in \mathscr{B}$.

Clearly, $\omega_F(x) := (\omega(xF))^{-1} \omega(xy)$ is the conditional probability that x occurs (upon execution of some E containing x), given that the operation F is (has been, is to be) executed. Thus, ω is a state on $\overrightarrow{\mathscr{A}\mathscr{B}}$ iff the probability of securing a given outcome of $\mathscr{A}$ is independent of which operation in $\mathscr{B}$ is executed. We say that such a weight *exhibits no influence* of $\mathscr{B}$ on $\mathscr{A}$.

One defines compound operations "in the other direction" by introducing a map $\pi: YX \to XY$ given by $\pi(yx) = xy$, and then letting

$\overleftarrow{\mathscr{A}\mathscr{B}} = \pi(\overrightarrow{\mathscr{B}\mathscr{A}})$. States on $\overleftarrow{\mathscr{A}\mathscr{B}}$ are states on $\mathscr{A} \times \mathscr{B}$ displaying no influence of $\mathscr{A}$ on $\mathscr{B}$. The *pre-tensor product* of $\mathscr{A}$ and $\mathscr{B}$ is the quasimanual

$$\mathscr{A}\mathscr{B} = \overrightarrow{\mathscr{A}\mathscr{B}} \cup \overleftarrow{\mathscr{A}\mathscr{B}}$$

Evidently, $\mathscr{A} \times \mathscr{B} \subseteq \mathscr{A}\mathscr{B}$ and $\Omega(\mathscr{A}\mathscr{B})$ consists exactly of those states ω on $\mathscr{A} \times \mathscr{B}$ that exhibit no influence of either of $\mathscr{A}$ and $\mathscr{B}$ on the other.

Lemma 3.2. $V_+(\mathscr{A}\mathscr{B})$ consists of those nonnegative functions ω on XY such that (i) ω is summable over every $\mathscr{A} \times \mathscr{B}$ operation, and (ii) $\omega(\cdot\, y)$ and $\omega(x\, \cdot)$ are $\mathscr{A}$- and $\mathscr{B}$-weights, respectively, for every $y \in Y$ and every $x \in X$.

Proof. Since $\mathscr{A} \times \mathscr{B} \subseteq \mathscr{A}\mathscr{B}$, every positive weight on the latter is summable over product operations. If ω is such a weight and $x \in X$ is fixed, $\omega(x\, \cdot)$ is a positive weight on $\mathscr{B}$ by Lemma 3.1; similarly, $\omega(\cdot\, y)$ is a positive weight on $\mathscr{A}$ for every fixed $y \in Y$. Conversely, suppose $\omega : XY \to \mathbf{R}$ is nonnegative and summable over elements of $\mathscr{A} \times \mathscr{B}$. Then for all $E \in \mathscr{A}$, $F \in \mathscr{B}$,

$$\sum_{xy \in EF} \omega(xy) = \sum_{x \in E} \left[\sum_{y \in F} \omega(xy) \right] = \sum_{y \in F} \left[\sum_{x \in E} \omega(xy) \right]$$

If ω is a weight in each variable separately, the sum is independent of both E and F, i.e., ω is a nonnegative $\mathscr{A} \times \mathscr{B}$-weight. By Lemma 3.1, $\omega \in V_+(\mathscr{A}\mathscr{B})$. ■

In general, the pre-tensor product $\mathscr{A}\mathscr{B}$ of manuals $\mathscr{A}$ and $\mathscr{B}$ is not a manual, nor even a premanual. On the other hand, if $\mathscr{A}$ and $\mathscr{B}$ are unital, then so is $\mathscr{A}\mathscr{B}$, and we may therefore form the *tensor product* $\mathscr{A} \otimes \mathscr{B} = \langle \mathscr{A}\mathscr{B} \rangle$. Notice that $\Omega(\mathscr{A} \otimes \mathscr{B}) = \Omega(\mathscr{A}\mathscr{B})$, hence, $V(\mathscr{A} \otimes \mathscr{B}) = V(\mathscr{A}\mathscr{B})$.

Given a pair of unital orthoalgebras Π_1 and Π_2, one defines their tensor product as follows (Lock, 1981): Let $\mathscr{A}$ be the manual of finite partitions of the unit in Π_1, $\mathscr{B}$ that of finite partitions of the unit in Π_2, and (since these manuals are unital) form $\mathscr{A} \otimes \mathscr{B}$. One then defines $\Pi_1 \otimes \Pi_2 := \Pi(\mathscr{A} \otimes \mathscr{B})$. As a consequence of Lemma 3.2, one has the following result.

Theorem 3.3. The state space of a tensor product $\Pi_1 \otimes \Pi_2$ of orthoalgebras is the set of states on $\Pi_1 \times \Pi_2$ whose marginals are measures.

Given two measurable spaces (S, Σ) and (T, Ξ), one may form the tensor product of their affiliated partition manuals and compare this to be partition manual of the field product $(S \times T, \Sigma \otimes \Xi)$. Lock (1981) has shown that the logic of the former is a Boolean algebra isomorphic, not to $\Sigma \otimes \Xi$, but rather to the clopen field of the product of the Stone spaces of Σ and Ξ. On the other hand, as pointed out in Randall and Foulis (1981), the state spaces of these manuals are essentially the same. When we deal with measure algebras matters become more complicated: Certainly, every measure on the field product of two measure algebras gives rise to a measure on the corresponding orthoalgebraic tensor product, but, as the following example shows, there will in general be measures on the latter not corresponding to any measure on the former.

Example 3.4. Let μ be Lebesgue measure on $[0, 1]$, let Σ be the algebra of Lebesgue measurable subsets of $[0, 1]$, and let Σ/μ be the associated measure algebra. Let D be the diagonal of $[0, 1] \times [0, 1]$ and let λ denote the linear Lebesgue measure on D. Let ω be the function on $\Sigma \times \Sigma$ given by $\omega(A, B) = \lambda((A \times B) \cap D)$. Since $\omega(\cdot, \cdot)$ is a μ-absolutely continuous measure in each variable separately, ω is well-defined on $\Sigma/\mu \times \Sigma/\mu$, and extends uniquely to a state on the *orthoalgebraic* tensor product of Σ/μ with itself. However, as ω is not absolutely continuous with respect to two-dimensional Lebesgue measure, ω does not lift to the field product $\Sigma/\mu \otimes \Sigma/\mu$.

Lemma 3.2 can be "linearized" as follows:

Lemma 3.5. $V_+(\mathscr{A}\mathscr{B})$ is affinely isomorphic to the set of separately weak*-continuous, positive bilinear forms on $V^*(\mathscr{A}) \times V^*(\mathscr{B})$.

Proof. Let $\omega \in V_+(\mathscr{A}\mathscr{B})$. If $g \in V^*(\mathscr{B})$, define $\hat{\omega}(g)(x) = g(\omega(x, \cdot))$; clearly, $\hat{\omega}(g)$ belongs to $V_+(\mathscr{A})$. Hence, define $\Phi_\omega(f, g) = f(\hat{\omega}(g))$ for every $f \in V^*(\mathscr{A})$ and $g \in V^*(\mathscr{B})$. Now, Φ is bilinear; by construction, it is weak*-continuous in f for fixed g. Since the construction is essentially symmetric in f and g, Φ_ω is also weak*- continuous in g for fixed f. The map $\omega \mapsto \Phi_\omega$ is clearly affine, and, since Φ is separately weak*-continuous, an injection. To see that it is a surjection, define, for a given separately weak*-continuous positive bilinear form Φ, a function $\omega(x, y) = \Phi(f_x, f_y)$. The continuity assumption ensures both that ω is summable over $\mathscr{A} \times \mathscr{B}$ operations and that $\omega(x\,\cdot)$ and $\omega(\cdot\, y)$ are weights. By Lemma 3.3, ω is a positive weight on $\mathscr{A} \otimes \mathscr{B}$; evidently, $\Phi = \Phi_\omega$. ■

If V and W are normed spaces, let $B(V, W)$ denote the space of bounded bilinear forms on $V \times W$. If V and W are arbitrary base-normed spaces, let $V * W$ denote the subspace of $B(V^*, W^*)$ generated by forms $\Phi \geq 0$ that are weak*-continuous in each variable separately. Order $V * W$ by the cone of positive separately weak*-continuous forms, and take as a base for this cone the set

$$\Omega = \{\Phi \in V * W \mid \Phi(e_1, e_2) = 1\}$$

The space $V * W$ is canonically isomorphic to the space of regular weak*-to-weakly continuous operators from V^* to W. We will confuse the two spaces, identifying forms with operators and vice versa without further comment.

Theorem 3.6. $V(\mathscr{A}\mathscr{B})$ is isomorphic as a base-normed space to $V(\mathscr{A}) * V(\mathscr{B})$.

Proof. This follows from Lemma 3.5 and the construction of $V * W$, since cone-base spaces having affinely isomorphic cone-bases are isomorphic. ■

In particular, that $V(\mathscr{A}) * W(\mathscr{B})$ is complete follows from the completeness of $V(\mathscr{A}\mathscr{B})$. We digress to show that $V * W$ is complete for arbitrary complete base-normed spaces V and W.

If (V, B) is a *complete* base-normed space, then its cone-base B is σ-convex in the sense that, for any sequence of coefficients $t_i \geq 0$ with $\sum_i t_i = 1$, and for any sequence v_i in B, the convex series $\sum_i t_i v_i$ is norm-convergent to an element of B. Conversely, it can be shown (Rüttimann and Schindler, 1987, Theorem 2.8) that if there is any Hausdorff linear topology τ on B such that every convex series in B is τ-convergent to an element of B, then V is complete.

Lemma 3.7. Let V and W be complete base-normed spaces. Then $V * W$ is a complete base-normed space. The base norm is no smaller than the norm corresponding to the operator norm.

Proof. That Ω is a cone-base is clear; and $V * W$ is positively generated by definition. The operator norm of a positive element f of $V * W$ is $\| f(e) \|$—i.e., coincides with its base norm. Hence, the base-norm unit ball of $V * W$ is at least as small as the unit ball of $B^+(V, W)$, i.e., the base norm is no smaller than the operator norm. To see that $V * W$ is complete, we will show that its base Ω is σ-convex with respect to the topology corresponding to the pointwise convergence of bilinear forms on $[0, e_1] \times [0, e_2]$. If $\Phi_i \in \Omega$ and t_i are convex coefficients, then for every $f \in [0, e_1]$ and $g \in [0, e_2]$, $\sum_i t_i \Phi_i(f, g)$ is a convex, hence convergent, series in the real interval $[0, 1]$; thus $\sum_i t_i \Phi_i$ converges pointwise on $[0, e_1] \times [0, e_2]$ to a bounded biaffine function Φ on $[0, e_1] \times [0, e_2]$, which then lifts to a bounded bilinear functional on $V^* \times W^*$. Now, for fixed f, the series $\sum_i t_i \Phi(f, \cdot)$ is a convex series in the positive part of W's unit ball, hence converges in norm to an element of W. It follows that $\Phi(f, \cdot)$ is weak-* continuous in the second argument for any $f \in V$. Likewise, $\Phi(\cdot, g)$ is weak-* continuous for all $g \in W$. ■

If V is finite-dimensional, then every linear operator $V^* \to W$ is weak*-to-weakly continuous. Thus we have the following strengthening of Theorem 1.1:

Corollary 3.8. If either of $V(\mathscr{A})$ or $V(\mathscr{B})$ is finite-dimensional, then

$$V(\mathscr{A}\mathscr{B}) \simeq V(\mathscr{A}) \otimes V(\mathscr{B})$$

As a further application of Theorem 3.6, we will show (Theorem 4.2) that the tensor product of discrete manuals is again discrete.

Kläy, Randall, and Foulis applied their theorem to characterize the states on a tensor product of finite-dimensional complex frame manuals as the trace-1 operators W on the tensor product of the associated Hilbert spaces satisfying the condition of *positivity on pure tensors*: $\langle W\phi \otimes \psi, \phi \otimes \psi \rangle \geq 0$ for all vectors ϕ, ψ. Elsewhere (Wilce, 1990) the authors used Theorem 3.6 to extend this result as follows:

Example 3.9. Let $\mathscr{A}$ and $\mathscr{B}$ be the frame manuals of two separable complex Hilbert spaces $\mathbf{H}$ and $\mathbf{K}$, respectively (each of dimension at least 3). Let $\mathscr{B}_h$ and $\mathscr{B}_{1,h}$ denote, respectively, the space of bounded self-adjoint operators on $\mathbf{H}$ and the self-adjoint trace-class of $\mathbf{H}$. Then $V_+(\mathscr{A}\mathscr{B})$ is isomorphic to the space of weak*-to-weakly continuous positive linear maps

$$\mathscr{B}_h(\mathbf{H}) \to \mathscr{B}_{1,h}(\mathbf{K})$$

Such a map gives rise to a linear operator

$$\Lambda : \mathbf{H} \otimes \bar{\mathbf{H}} \to \mathbf{K} \otimes \bar{\mathbf{K}}$$

which satisfies the positivity condition $(\Lambda x \otimes \bar{x}, y \otimes \bar{y}) \geq 0$ for all $x \in \mathbf{H}$ and all $y \in \mathbf{K}$. If it is Hilbert–Schmidt, Λ gives rise in turn to an operator W on $\mathbf{H} \otimes \mathbf{K}$ such that $(Wx \otimes y, x \otimes y) \geq 0$ for all $x \in \mathbf{H}$ and $y \in \mathbf{K}$. Conversely, any Hilbert–Schmidt operator on $\mathbf{H} \otimes \mathbf{K}$ satisfying the indicated positivity condition and having trace-class marginals gives rise to a Hilbert–Schmidt operator Λ as above. In particular, every element of the quantum mechanical state space, i.e., every density operator on $\mathbf{H} \otimes \mathbf{K}$, corresponds to an element of $\Omega(\mathscr{A}\mathscr{B})$. [Even in the finite-dimensional case, however, the inclusion is proper (cf. Kläy *et al.*, 1987)].

4. TENSOR PRODUCTS OF SIGNED WEIGHT SPACES

In this section we consider the spaces $V(\mathscr{A}) \otimes V(\mathscr{B})$, as embedded in $V(\mathscr{A}\mathscr{B})$, and $L_o(\mathscr{A}) \otimes L_o(\mathscr{B})$, as embedded in $L_o(\mathscr{A}\mathscr{B})$. We begin with a review of ordered tensor products, following the survey paper of Wittstock (1974), to which we direct the reader for additional information. We require the notion of a *regularly* ordered normed space: An ordered normed space V is *regular* iff $-u \leq v \leq u$ implies $\|v\| \leq \|u\|$ for all $u, v \in V$, and $\|u\| = \inf\{\|v\| \,|\, v \geq u, -u\}$. It is not difficult to see that any base-normed or order-unit normed space is regular. More generally, it is a standard result that any positively generated normed space whose norm is monotone possesses an equivalent regular norm.

Let V and W be regularly ordered normed spaces. We may represent the algebraic tensor product $V \otimes W$ either as a space of bilinear forms or as a space whose dual consists of bilinear forms. That is, we may embed $V \otimes W$ in $B^+(V^*, W^*)$ or in $B^+(V, W)^*$. The order structures associated with these two representations are different: As a space of bilinear forms, $V \otimes W$ is ordered by the cone C_i consisting of all $\tau \in V \otimes W$ such that $(\tau, f \otimes g) \geq 0$ for all positive functionals $f \in V^*$ and $g \in W^*$. The cone C_i is closed with respect to the norm inherited from $B(V, W)$. A standard argument using the Hahn–Banach theorem shows that the positive cone of $V \otimes W$ as embedded in $B(V, W)^*$ is the closed convex hull $C_p := \overline{\text{con}}(V_+ \otimes W_+)$ of the set of pure tensors of positive elements of V and W [the closure taken with respect to the norm inherited from $B(V, W)^*$]. The norm of $B(V, W)^*$ and that of $B(V^*, W^*)$ are monotone on the cones C_p and C_i, respectively, so one may form the associated regular norms $\|\cdot\|_p$ and $\|\cdot\|_i$. The resulting (regular) ordered normed spaces $(V \otimes W, C_p, \|\cdot\|_p)$ and $(V \otimes W, C_i, \|\cdot\|_i)$ are the *projective* and *injective* ordered tensor products of V and W. We denote them by $V \otimes_p W$ and $V \otimes_i W$, respectively.

If V and W are order-unit normed spaces having respective order units e_1 and e_2, then $V \otimes_p W$ and $V \otimes_i W$ are both order-unit normed with order unit $e = e_1 \otimes e_2$. If V and W are both base-normed, with respective dual order units e_1 and e_2, then both $V \otimes_p W$ and $V \otimes_i W$ are base-normed, with bases $e^{-1}(1) \cap C_i$ and $e^{-1}(1) \cap C_p$. [Here e is the unique functional on $V \otimes W$ determined by the condition $e(v, w) = e_1(v)e_2(w)$.]

It will be important that, for any (regular) ordered normed spaces V, W, and B, any bounded positive bilinear form $\Phi: V \times W \to B$ remains bounded and positive as a linear map $\phi: V \otimes_p W \to B$. In particular, $(V \otimes_p W)^*$ is isomorphic to the positively-generated part $B^+(V, W)$ of the space of bounded bilinear forms on $V \times W$. Also important is the observation that, if V and W arise as spaces of functions—say, as subspaces of $\mathbf{R}^X$ and $\mathbf{R}^Y$, respectively, ordered pointwise—then the pointwise order on $V \otimes W$ as a subspace of $\mathbf{R}^{XY}$ coincides with that of $V \otimes_i W$, regardless of the particular representation of V and W as function spaces.

Lemma 4.1. The image of $V(\mathscr{A}) \otimes V(\mathscr{B})$ in $V(\mathscr{A}\mathscr{B})$ is linearly and order-isomorphic $V \otimes_i W$. This last is pointwise dense in $V(\mathscr{A}\mathscr{B})$.

Proof. The first statement is immediate from the preceding remark. Since pure tensors separate the points of $V^*(\mathscr{A}) \otimes V^*(\mathscr{B})$, the algebraic tensor product is weak* dense in $(V^*(\mathscr{A}) \otimes_p V^*(\mathscr{B}))^*$. This last is isometrically isomorphic to positively generated part of the space of bounded bilinear forms on $V^*(\mathscr{A}) \times V^*(\mathscr{B})$, hence, by Lemma 3.6, contains $V(\mathscr{A}\mathscr{B})$. The density claim now follows. ■

Ideally, one would like to obtain a density result for $V(\mathscr{A}) \otimes V(\mathscr{B})$ in terms of the *ordered* injective tensor norm. Even in the case that both factors are of the form $L_1(\mu)$, this is not possible: Again, the weight ω of Example 3.5 provides the counterexample. Having absolutely continuous marginals, ω defines an element of $V(\mathscr{A}\mathscr{B})$; however (Wittstock, 1974, Theorems 4.5 and 4.7),

$$L_1(\mu) \hat{\otimes}_i L_1(\mu) \simeq L_1(\mu) \hat{\otimes}_\pi L_1(\mu) \simeq L_1(\mu \times \mu)$$

The latter contains no element corresponding to ω.

We come now to the principal result of this paper:

Theorem 4.2. If $\mathscr{A}$ and $\mathscr{B}$ are discrete, then $\mathscr{A}\mathscr{B}$ is also discrete. Moreover, $L_o(\mathscr{A}\mathscr{B})$ is norm- and order-isomorphic to $L_o(\mathscr{A}) \otimes_p L_o(\mathscr{B})$.

Proof. Let $\phi: V(\mathscr{A}\mathscr{B}) \to L_o^*(\mathscr{A}\mathscr{B})$ be the bounded, positive injection given by $\phi(\omega)(f) = f(\omega)$ $(\omega \in V, f \in L_o)$. We will show that ϕ is a surjection by exhibiting its inverse.

The representation of $V(\mathscr{A}\mathscr{B})$ as $V(\mathscr{A}) * V(\mathscr{B})$ (Theorem 3.6) yields a bounded bilinear map

$$\psi: V^*(\mathscr{A}) \times V^*(\mathscr{B}) \to V^*(\mathscr{A}\mathscr{B})$$

given by $\psi(f, g)(\omega) = \Phi_\omega(f, g)$, Φ_ω being the bilinear form representing the signed weight ω. The restriction of ψ to $L_o(\mathscr{A}) \times L_o(\mathscr{B})$ is bounded, and hence lifts to a bounded—and, clearly, injective—linear map

$$\psi: L_o(\mathscr{A}) \otimes_p L_o(\mathscr{B}) \to V^*(\mathscr{A}\mathscr{B})$$

If $f \in L_o(\mathscr{A})$ and $g \in L_o(\mathscr{B})$ are given by $f = \sum_{A \in I} \alpha_A f_A$ and $g = \sum_{B \in J} \beta_B f_B$ for some finite collections I and J of events, then

$$\psi(f, g) = \sum_{A \in I, B \in J} \alpha_A \beta_B f_{AB}$$

Thus, the range of ψ is contained in $L_o(\mathscr{A}\mathscr{B})$. Since $f_{xy} = \psi(f_x, f_y)$ for all $x \in X(\mathscr{A})$, $y \in X(\mathscr{B})$, ψ's range in fact equals $L_o(\mathscr{A}\mathscr{B})$. Since ψ is bounded and positive as a bilinear form, it remains positive on the projective tensor cone. It follows now that we have a bounded, positive bijection

$$\psi^*: \quad L_o^*(\mathscr{A}\mathscr{B}) \to B^+(L_o(\mathscr{A}), L_o(\mathscr{B}))$$

Now if Φ is a positive bounded bilinear form in $B^+(L_o(\mathscr{A}), L_o(\mathscr{B}))$, define a function $\omega(xy) = \Phi(f_x, f_y)$ on XY, and notice that, since $\mathscr{A}$ and $\mathscr{B}$ are discrete, this function is a positive weight in each variable separately. For any finite events A of $\mathscr{A}$ and B of $\mathscr{B}$, $\sum_{xy \in AB} \omega(xy) = \Phi(f_A, f_B) \le \|\Phi\| < \infty$.

Since the net of finite rectangular events AB is cofinal in the net of all finite events of $\mathscr{A}\times\mathscr{B}$, ω is summable over $\mathscr{A}\mathscr{B}$. Thus, by Lemma 3.2, ω is a positive weight on $\mathscr{A}\mathscr{B}$. Notice that $\omega=0$ iff $\Phi=0$. Thus, we may regard ψ^* as a bounded, positive injection $L_o^*(\mathscr{A}\mathscr{B})\to V(\mathscr{A}\mathscr{B})$. Evidently, for any fixed ω and outcomes x and y, we have

$$\psi^*(\phi(\omega))(xy)=\phi(\omega)(\psi(f_x\otimes f_y))=f_{xy}(\omega)=\omega(xy)$$

whence $\psi^*\circ\phi$ is the identity on $V(\mathscr{A}\mathscr{B})$. Similarly, $\phi\circ\psi^*$ is the identity on $L_o^*(\mathscr{A}\mathscr{B})$. It follows that $\mathscr{A}\mathscr{B}$ is discrete. This also establishes that $V(\mathscr{A}\mathscr{B})$ is isomorphic to the positively generated part of the space of bounded bilinear forms on $L_o(\mathscr{A})\times L_o(\mathscr{B})$, from which we conclude that ψ is an order isomorphism: If $\psi(\tau)\geq 0$ for some tensor τ, then

$$0\leq\psi(\tau)(\omega)=\psi^*(\omega)(\tau)$$

for all positive weights $\omega\in V_+$. Hence, for all positive bilinear forms Φ, $\Phi(\tau)\geq 0$, whence $\tau\in C_p$. Finally, since the projective tensor norm on $L_o\otimes_p L_o$ is exactly that induced by the base norm on $B^+(L_o, L_o)$, ψ is an isometry. ■

We conclude with some remarks concerning the state space of $\mathscr{A}\mathscr{B}$. If Γ and K are the respective cone-bases of base-normed spaces V and W, denote the cone-bases of $V\otimes_p W$ and $V\otimes_i W$ by $\Gamma\otimes_p K$ and $\Gamma\otimes_i K$, of Γ and K. Notice that if $V(\mathscr{A})$ and $V(\mathscr{B})$ are finite-dimensional, then, by Corollary 3.8,

$$\Omega(\mathscr{A}\mathscr{B})\simeq\Omega(\mathscr{A})\otimes_i\Omega(\mathscr{B})$$

Let $\Gamma=\Omega(\mathscr{A})$ and $K=\Omega(\mathscr{B})$. If $\mathscr{A}$ and $\mathscr{B}$ are locally finite, Γ and K are compact by Lemma 2.1. Namioka and Phelps introduce two tensor products for compact convex sets, as follows: Recall that any compact convex set Γ may be identified with the state space of the order-unit space $A(\Gamma)$ of continuous affine functionals on Γ. The (compact) convex sets $\Gamma\,\square\,K$ and $\Gamma\,\triangle\,K$ are defined to be the state spaces, respectively, of the order-unit spaces $A(\Gamma)\otimes_p A(K)$ and $A(\Gamma)\otimes_i A(K)$. Since $\Gamma=\Omega(\mathscr{A})$ and $K=\Omega(\mathscr{A})$ are the state spaces of the order-unit spaces $L_o(\mathscr{A})$ and $L_o(\mathscr{B})$, it follows that $\bar{L}_o(\mathscr{A})=A(\Gamma)$ and $\bar{L}_o(\mathscr{B})=A(K)$. Thus, by Theorem 4.2, the state space $\Omega(\mathscr{A}\mathscr{B})$ is $\Omega(\mathscr{A})\,\square\,\Omega(\mathscr{B})$. It is known that $\Gamma\,\square\,K$ is a simplex if Γ and K are simplices. Theorem 4.2 allows us to obtain a slightly stronger result for (not necessarily compact) state spaces of discrete quasimanuals.

Theorem 4.3. If $\mathscr{A}$ and $\mathscr{B}$ are discrete and both $\Omega(\mathscr{A})$ and $\Omega(\mathscr{B})$ are simplices, then $\Omega(\mathscr{A}\mathscr{B})$ is also a simplex.

Proof. It is sufficient to show that $V(\mathscr{A}\mathscr{B})$ is a lattice. A theorem of Davies (1968) asserts that the dual of an ordered Banach space L is a lattice iff L is regular and enjoys the Riesz interpolation property. Observe that, as the spaces L_o and $\bar{L}_o$ have monotone norms, they can be equivalently renormed so as to be regular (obviously, this has no effect on their order structure, nor on that of their duals). Since $V(\mathscr{A})$ and $V(\mathscr{B})$ are lattices, both $\bar{L}_o(\mathscr{A})$ and $\bar{L}_0(\mathscr{B})$ have the Riesz interpolation property. By Wittstock (1974, Theorem 3.3), the projective ordered tensor product $\bar{L}_o(\mathscr{A}) \otimes_p \bar{L}_o(\mathscr{B})$ is a regular ordered space having the Riesz interpolation property. Its dual $B^+(L_o(\mathscr{A}), L_o(\mathscr{B}))$, which Theorem 4.2 identifies as $V(\mathscr{A}\mathscr{B})$, is therefore a lattice. ■

If $\Omega(\mathscr{A}\mathscr{B})$ is to serve as a model of two separated systems described individually by $\Omega(\mathscr{A})$ and $\Omega(\mathscr{B})$, then it is important to characterize the extreme points of $\Omega(\mathscr{A}\mathscr{B})$ in terms of those of $\Omega(\mathscr{A})$ and $\Omega(\mathscr{B})$. As observed by Kläy *et al.*, Corollary 3.8 implies that

$$\Omega(\mathscr{A}\mathscr{B}) \subseteq \mathrm{Aff}(\partial\Omega(\mathscr{A}) \otimes \partial\Omega(\mathscr{B}))$$

[This provides, incidentally, a negative answer to the question raised by Namioka and Phelps (1969), of whether the projective tensor product $\Gamma \triangle K$ of two compact convex sets need be a face of $\Gamma \square K$.] Even in the finite-dimensional case, we have no real understanding of which affine combinations of pure tensors of extreme weights are extreme—indeed, this is a problem of long standing concerning the $\square$ product of the tensor product of compact convex sets. The situation is already very involved when $\mathscr{A}$ and $\mathscr{B}$ are finite-dimensional frame manuals: If $\mathscr{A} = \mathscr{B} = \mathscr{F}(\mathbf{C}^n)$, then (by Corollary 3.8), the problem amounts to that of characterizing the set of positive linear operators $\phi: M_n \to M_n$ on the algebra of $n \times n$ complex matrices such that $\phi(\mathbf{1}_n)$ has unit trace. This is nontrivial even when $n=2$ (Choi, 1975).

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